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ANALOGUES OF THE "ZERO-TWO" LAW FOR POSITIVE LINEAR CONTRACTIONS IN L^p AND $C(X)$

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ABSTRACT

Let T be a positive linear contraction in L^p $(1 \leqq p < \infty)$, then we show that

$$\lim \| T^p f - T^{n+1} f \|_p \leqq (1-\varepsilon) 2^{1/p} \qquad (f \in L_+^p,\ \varepsilon > 0 \text{ independent of } f)$$

implies already $\lim_{n\to\infty} \| T^n f - T^{n+1} f \|_p = 0$. Several other related results as well as uniform variants of these are also given. Finally some similar results in L^∞ and $C(X)$ are shown.

Introduction

Let T be a positive linear contraction in L^1. In [8] D. Ornstein and L. Sucheston showed that

$$\sup_{\|f\|_1 \leq 1} \lim_{n\to\infty} \| T^n f - T^{n+1} f \|_1 \tag{1}$$

is either 0 or 2. This surprising result opened a new direction of research. We mention only the work of Derrienic [1], Foguel [2], and Lin [6]. These papers deal mainly with Markov operators on L^∞ but by duality both types of result are equivalent. Derrienic [1] gives a probabilistic characterization of the 0–2 law in terms of the tail σ-algebra of the associated Markov chain. Lin [6] gives a decomposition of the underlying measure space in a "zero" and a "two" part. Interchanging sup and lim in (1) the same was shown by Foguel [2]. Results of the latter type will be called uniform "zero–two" laws. Usually the proof of a uniform result is very similar, but sometimes also simpler than the corresponding non-uniform result. This is the reason why we omit often many details in the proof of the uniform results.

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Very recently Zaharopol [11] generalized the uniform variant of (1) to L^p-spaces ($1 \leqq p < \infty, p \neq 2$). Katznelson and Tzafriri [4] proved the same for all $1 \leqq p < \infty$ and gave also a simpler proof of Zaharopol's theorem. The paper of Katznelson and Tzafriri is the only one which does not use the key idea of [8]. Zaharopol's theorem involves the modulus $|T^n - T^{n+1}|$ of the operator $T^n - T^{n+1}$ (cf. Schaefer [10], p. 229):

$$\lim_{n\to\infty} \sup_{\|f\|_p \leqq 1} \| \, |T^n - T^{n+1}| f \|_p < 2 \Rightarrow \lim_{n\to\infty} ||| T^n - T^{n+1} ||| = 0.$$

For $p = 1$ we have $||| \, |T^n - T^{n+1}| \, |||_p = ||| T^n - T^{n+1} |||_p$ but not for $p > 1$.

In this paper we study (1) and related expressions when the L^1-norm is replaced by L^p-norms (section 1) or a supremum norm (section 2). The Theorem of Zaharopol–Katznelson–Tzafriri will turn out to be a corollary.

The following constants play a crucial role:

$$\alpha_p := \sup_{0 \leqq x \leqq y} \left(\frac{(1+x)^p + (1+y)^p + (y-x)^p}{1 + x^p + y^p} \right)^{1/p} \qquad (1 \leqq p < \infty).$$

Much is known about α_p (cf. 1.4). For instance, we have $\alpha_1 = 2$, $\alpha_2 = \sqrt{3}$ and $\sqrt{3} \leqq \alpha_p < 2$ ($p \neq 1$). We now have (cf. Theorem 1.1) for any positive linear contraction T in L^p

$$\sup_{\|f\|_p \leqq 1} \lim_{n\to\infty} \| T^n - T^{n+1} f \|_p \quad \text{is either} \geqq \alpha_p \text{ or } 0.$$

This result is sharp, i.e. α_p may not be replaced by a larger number (cf. Example 1.3) and we may also interchange sup and lim (cf. Theorem 1.7). A more beautiful result can be proved if the supremum is only taken over non-negative f:

$$\sup_{\|f\|_p \leqq 1, f \geqq 0} \lim_{n\to\infty} \| T^n f - T^{n+1} f \|_p \quad \text{is either } 2^{1/p} \text{ or } 0.$$

In the "zero" case one can show that

$$\lim_{n\to\infty} \left\| g - \frac{1}{n+1} \sum_{i=0}^{n} T^i f \right\|_p = 0$$

implies already (Corollary 1.6)

$$\lim_{n\to\infty} \| g - T^n f \|_p = 0.$$

The last fact was observed independently by Zaharopol (private communication). Moreover, his argument is constructive unlike ours. For $p = 1$ special cases of this result were proved by Orey [7] and Ornstein and Sucheston [8] (see also Greiner and Nagel [3]).

Very strong variants of the above results are given when T is also a compact operator (cf. Corollary 1.8).

The proof of Ornstein and Sucheston is based on the representation

$$T^m f = (I + T)^n u + v \qquad (\| u \| \leqq 2^{-n}, \ \| v \| \text{ small})$$

and the cancellation

$$\begin{aligned} \| (I - T)T^m f \| &= \| (I - T)(I + T)^n u + v \| \\ &\leqq 2^{1-n} + \| v \| + \sum_{i=1}^{n} \left| \binom{n}{i} - \binom{n}{i-1} \right| 2^{-n} \\ &\underset{\substack{n \to \infty \\ \| v \| \to 0}}{\longrightarrow} 0. \end{aligned}$$

This method works with minor changes also in the case L^∞ and $C(X)$. This is done is Section 2.

In the L^p-case ($1 < p < \infty$) their method doesn't work any more. In this case we use the more difficult representation

$$T^m f = \sum_{i=0}^{n-1} T^i u + v \qquad (\| u \|_p \leqq n^{-1/p}, \ \| v \|_p \text{ small})$$

and the stronger cancellation

$$\| (I - T)T^m f \|_p = \| u - T^n u + (I - T)v \|_p \leqq 2n^{-1/p} + 2 \| v \|_p \underset{\substack{n \to \infty \\ \| v \|_p \to 0}}{\longrightarrow} 0.$$

1. Positive contractions in L^p ($1 \leqq p < \infty$)

In the sequel (Ω, A, μ) will be a σ-finite measure space and T will be a positive linear contraction in $L^p = L^p(\Omega, A, \mu)$ where $1 \leqq p < \infty$ will be fixed. We denote $L^p_+ := \{f \in L^p : f \geqq 0\}$. For $f \in L^p$, f^+ denotes $\sup(f, 0)$.

1.1. Theorem. *The following conditions are equivalent*:

(i) $\lim_{n \to \infty} \| T^n f - T^{n+1} f \|_p = 0$ ($f \in L^p$).

(ii) *There exists* $\varepsilon > 0$ *such that*

$$\lim_{n \to \infty} \| T^n f - T^{n+1} f \|_p \leqq 2^{1/p}(1 - \varepsilon) \qquad (f \in L^p_+, \ \| f \|_p \leqq 1).$$

(iii) *There exists* $\varepsilon > 0$ *such that*

$$\lim_{n\to\infty} \| T^n f - T^{n+1} f \|_p \leqq (1-\varepsilon)\alpha_p \qquad (f \in L^p,\ \| f \|_p \leqq 1).$$

(iv) *There exists* $\varepsilon > 0$ *such that*

$$\inf_{n\to\infty} \| (T^n f - T^{n+1} f)^+ \|_p \leqq (1-\varepsilon) \qquad (f \in L^p_+,\ \| f \|_p \leqq 1).$$

PROOF. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are obvious. Assume now that (ii) holds and let $f \in L^p_+$, $\| f \|_p \leqq 1$ be given. In order to prove (iv) we have to find an $\varepsilon' > 0$ (independent of f) and $n \in \mathbf{N}$ such that

$$(1) \qquad \| (T^n f - T^{n+1} f)^+ \|_p \leqq 1 - \varepsilon'.$$

Since $(\| T^n f \|_p)_{n\in\mathbf{N}}$ is decreasing $\alpha := \lim_{n\to\infty} \| T^n f \|_p \leqq 1$ exists. By (ii) there exists $n \in \mathbf{N}$ such that ($\delta > 0$ will be specified later)

$$\| T^n f - T^{n+1} f \|_p \leqq (1+\delta)(1-\varepsilon)2^{1/p}, \qquad \| T^n f \|_p \leqq \alpha + \delta.$$

Because of

$$\| (T^n f - T^{n+1} f)^+ \|_p^p + \| (T^{n+1} f - T^n f)^+ \|_p^p = \| T^n f - T^{n+1} f \|_p^p \leqq (1+\delta)^p (1-\varepsilon)^p 2$$

we have either

$$(2) \qquad \| (T^n f - T^{n+1} f)^+ \|_p \leqq (1+\delta)(1-\varepsilon)$$

or

$$(3) \qquad \| (T^{n+1} f - T^n f)^+ \|_p \leqq (1+\delta)(1-\varepsilon).$$

From now on we assume that $\delta > 0$ was chosen so small that

$$(1+\delta)(1-\varepsilon) \leqq 1 - \varepsilon/2, \qquad \delta \leqq \varepsilon/8$$

and we set

$$\varepsilon' := \min(\varepsilon/8,\ 1 - (1 - (\varepsilon/4)^p)^{1/p}).$$

Then, in case (2), (1) obviously holds. Assume now that (3) holds. Denoting $u := \inf(T^n f, T^{n+1} f)$ we have

$$\begin{aligned} \| u \|_p &\geqq \| T^{n+1} f \|_p - \| T^{n+1} f - u \|_p \\ (4) \qquad &= \| T^{n+1} f \|_p - \| (T^{n+1} f - T^n f)^+ \|_p \\ &\geqq \alpha - (1 - \varepsilon/2). \end{aligned}$$

If $\alpha \leqq 1 - \varepsilon/4$ then

$$\| (T^n f - T^{n+1} f)^+ \|_p \leqq \| T^n f \|_p \leqq \alpha + \delta \leqq 1 - \varepsilon/8 \leqq 1 - \varepsilon'.$$

If $\alpha \geqq 1 - \varepsilon/4$ then (4) implies

$$| T^n f |^p = ((T^n f - T^{n+1} f)^+ + u)^p \geqq |(T^n f - T^{n+1} f)^+|^p + u^p$$

and we get

$$\| (T^n f - T^{n+1} f)^+ \|_p^p \leqq \| T^n f \|_p^p - \| u \|_p^p \leqq 1 - (\varepsilon/4)^p \leqq (1 - \varepsilon')^p.$$

Hence (1) holds in all cases for the above choice of ε'. Assume now that (iv) holds. Obviously it is enough to prove (i) for $f \in L_+^p$, $\| f \|_p \leqq 1$. Let $k \in \mathbf{N}$ be so large that

$$q := (1 + \tfrac{1}{2}(k+1)^{-p}(\varepsilon/2)^{kp})^{1/p} \leqq 1 + \varepsilon/2(1 - \varepsilon)$$

and denote

$$\alpha_0 := \lim \| T^n f \|_p.$$

We choose $n_1', n_1'' \in \mathbf{N}$ such that

$$\| T^{n_1'} f \|_p \leqq q \circ \alpha_0, \quad \| (T^{n_1''}(T^{n_1'} f) - T^{n_1''+1}(T^{n_1'} f))^+ \|_p \leqq (1 - \varepsilon) \| T^{n_1'} f \|_p.$$

Setting $n_1 := n_1' + n_1''$ we have

$$\| T^{n_1} f \|_p \leqq q \circ \alpha_0, \quad \| (T^{n_1} f - T^{n_1+1} f)^+ \|_p \leqq (1 - \varepsilon) q \alpha_0.$$

Now we define

$$u_1 := \inf(T^{n_1} f, T^{n_1+1} f), \quad v_1 := T^{n_1} f - u_1 = (T^{n_1} f - T^{n_1+1} f)^+.$$

Since $\| T^{n_1+n} f \|_p \geqq \alpha_0$ we have for any $n \in \mathbf{N}$

$$\| T^n u_1 \|_p \geqq \| T^{n_1+n} f \|_p - \| T^n v_1 \|_p \geqq \alpha_0 - (1 - \varepsilon) q \alpha_0 = \alpha_0 (1 - q + q\varepsilon) \geqq \tfrac{1}{2}\varepsilon \alpha_0.$$

Replacing f by u_1 and α_0 by $\alpha_1 := \lim_{n \to \infty} \| T^n u_1 \|_p \geqq \frac{1}{2}\varepsilon \alpha_0$ we can apply the above construction once more to find $n_2 \in \mathbf{N}$ such that $u_2 := \inf(T^{n_2} u_1, T^{n_2+1} u_1)$ satisfies

$$\alpha_2 := \lim_{n \to \infty} \| T^n u_2 \| \geqq \tfrac{1}{2}\varepsilon \alpha_1 \geqq (\tfrac{1}{2})^2 \alpha_0.$$

Continuing in this way until step k we find $n_1, \ldots, n_k$ such that the $u_0, \ldots, u_k$ defined by

$$u_0 := f, \quad u_i := \inf(T^{n_i} u_{i-1}, T^{n_i+1} u_{i-1})$$

satisfy

$$\lim_{n\to\infty} \| T^n u_i \| \geqq (\tfrac{1}{2}\varepsilon)^i \alpha_0 \qquad (0 \leqq i \leqq k). \tag{5}$$

Since T is positive it follows that

$$\begin{aligned} T^n u_i &\leqq \inf(T^{n_i+n} u_{i-1}, T^{n_i+n+1} u_{i-1}) \\ &\leqq \inf(T^{n_{i-1}+n_i+n} u_{i-2}, T^{n_{i-1}+n_i+n+1} u_{i-2}, T^{n_{i-1}+n_i+n+2} u_{i-2}). \end{aligned}$$

Continuing in this way we obtain

$$T^n u_k \leqq \inf\{T^{n_1+\cdots+n_k+n+j} f : 0 \leqq j \leqq k\}. \tag{6}$$

Now we define

$$\tilde{m} = n_1 + \cdots + n_k, \quad g_j := T^{\tilde{m}+j} f - u_k \qquad (0 \leqq j \leqq k).$$

Then we have

$$(k+1) T^{\tilde{m}+k} f = \sum_{j=0}^{k} T^{k-j}(u_k + g_j) = \sum_{i=0}^{k} T^i u_k + \sum_{j=0}^{k} T^{k-j} g_j.$$

Thus, if we denote

$$m_1 := \tilde{m} + k, \quad r_1 := \frac{1}{k+1} u_k, \quad s_1 := \frac{1}{k+1} \sum_{j=0}^{k} T^{k-j} g_j,$$

then we have the fundamental representation formula

$$T^{m_1} f = \sum_{i=0}^{k} T^i r_1 + s_1 \qquad (r_1, s_1 \in L_+^p).$$

Moreover,

$$r_1^p + s_1^p \leqq (r_1 + s_1)^p \leqq (T^{m_1} f)^p$$

implies

$$\begin{aligned} \| s_1 \|_p^p \leqq \| T^{m_1} f \|_p^p - \| r_1 \|_p^p &\leqq (q \cdot \alpha_0)^p - \left(\frac{1}{k+1} \| u_k \|_p \right)^p \\ &\leqq \left(1 + \tfrac{1}{2}(k+1)^{-p} \left(\frac{\varepsilon}{2} \right)^{kp} - (k+1)^{-p} \left(\frac{\varepsilon}{2} \right)^{kp} \right) \alpha_0^p. \end{aligned}$$

Setting $\gamma := (1 - \tfrac{1}{2}(k+1)^{-p}(\varepsilon/2)^{kp})^{1/p} < 1$ we have shown that

$$\| s_1 \|_p \leqq \gamma \alpha_0 \leqq \gamma \| f \|_p.$$

Replacing f by s_1 then by the same construction there exists $\tilde{m}_2 \in \mathbf{N}$ and $\tilde{r}_2, s_2 \in L^p_+$ such that

$$T^{\tilde{m}_2} s_1 = \sum_{i=0}^{k} T^i \tilde{r}_2 + s_2, \quad \| s_2 \|_p \leqq \gamma \| s_1 \| \leqq \gamma^2 \| f \|_p.$$

Defining now

$$m_2 := m_1 + \tilde{m}_2, \quad r_2 = \tilde{r}_2 + T^{\tilde{m}_2} r_1$$

we have

$$T^{m_2} f = T^{\tilde{m}_2 + m_1} f = \sum_{i=0}^{k} T^{\tilde{m}_2} T^i r_1 + \sum_{i=0}^{k} T^i \tilde{r}_2 + s_2 = \sum_{i=0}^{k} T^i r_2 + s_2.$$

Continuing in this way we obtain an increasing sequence (m_n) in $\mathbf{N}$ and sequences (r_n), (s_n) in L^p_+ such that

$$T^{m_n} f = \sum_{i=0}^{k} T^i r_n + s_n, \qquad \| s_n \|_p \leqq \gamma^n \| f \|_p.$$

Because of

$$\sum_{i=0}^{k} | T^i r_n |^p \leqq \left(\sum_{i=0}^{k} T^i r_n \right)^p \leqq | T^{m_n} f |^p$$

there exist $0 \leqq i_n \leqq k$ such that

$$\| T^{i_n} r_n \|_p^p \leqq \frac{1}{k+1} \| T^{m_n} f \|_p^p \leqq \frac{1}{k}.$$

Defining now

$$m_n^* := m_n + i_n, \quad r_n^* = T^{i_n} r_n, \quad s_n^* := T^{i_n} s_n$$

we obtain

$$T^{m_n^*} f = \sum_{i=0}^{k} T^i r_n^* + s_n^*, \quad \| r_n^* \|_p \leqq k^{-1/p}, \quad \| s_n^* \|_p \leqq \gamma^n.$$

Finally, we have

$$\begin{aligned} \| T^{m_n^*} f - T^{m_n^* + 1} f \|_p &\leqq \left\| (I - T) \left(\sum_{i=0}^{k} T^i r_n^* \right) \right\|_p + \| (I - T) s_n^* \|_p \\ &\leqq \| r_n^* - T^{k+1} r_n^* \|_p + 2\gamma^n \\ &\leqq 2k^{-1/p} + 2\gamma^n. \end{aligned}$$

Since $\gamma < 1$ and since $(\| T^n f \|_p)$ is decreasing the assertion (i) follows.

The proof is complete if we can show that (iii) implies (i). Surprisingly this part of the assertion lies rather deep. To show it we must use the implication (iv)$\Rightarrow$(i) twice.

We observe first that there exists $0 \leqq x \leqq y$ such that

$$\alpha_p = \left(\frac{(1+x)^p + (1+y)^p + (y-x)^p}{1+x^p+y^p}\right)^{1/p},$$

otherwise there would exist $0 \leqq x_n \leqq y_n$, $\lim y_n = \infty$ such that

$$\alpha_p = \lim_{n\to\infty} \left(\frac{(1+x_n)^p + (1+y_n)^p + (y_n-x_n)^p}{1+x_n^p+y_n^p}\right)^{1/p}.$$

But the last limit equals 1 contradicting

$$\alpha_p \geqq \left(\frac{(1+0)^p + (1+1)^p + (1-0)^p}{1+0^p+1^p}\right)^{1/p} = (1+2^{p-1})^{1/p} > 1.$$

Assume now that

$$\sup_{f\in L_+^p, \|f\|_p \leqq 1} \ \inf_{n\in\mathbf{N}} \min(\| (T^n f - T^{n+1} f)^+ \|_p, \| (T^n f - T^{n+2} f)^+ \|_p) = 1. \tag{7}$$

Then for any $0 < \varepsilon \leqq 1$ there exists $f_\varepsilon \in L_+^p$, $\| f_\varepsilon \|_p \leqq 1$ with

$$\| (T^n f_\varepsilon - T^{n+1} f_\varepsilon)^+ \|_p \geqq 1 - \varepsilon, \quad \| (T^n f_\varepsilon - T^{n+2} f_\varepsilon)^+ \|_p \geqq 1 - \varepsilon \quad (n \in \mathbf{N}).$$

Denoting

$$u_{n,\varepsilon} := \inf(T^n f_\varepsilon, T^{n+1} f_\varepsilon), \quad v_{n,\varepsilon} := \inf(T^n f_\varepsilon, T^{n+2} f_\varepsilon), \quad g_\varepsilon := y f_\varepsilon - T f_\varepsilon + x T^2 f_\varepsilon$$

we have for any $n \in \mathbf{N}$

$$u_{n,\varepsilon}^p + |(T^n f_\varepsilon - T^{n+1} f_\varepsilon)^+|^p \leqq (u_{n,\varepsilon} + (T^n f_\varepsilon - T^{n+1} f_\varepsilon)^+)^p = |T^n f_\varepsilon|^p,$$

$$\| u_{n,\varepsilon} \|_p^p \leqq \| T^n f_\varepsilon \|_p^p - \| (T^n f_\varepsilon - T^{n+1} f_\varepsilon)^+ \|_p^p \leqq 1 - (1-\varepsilon)^p \leqq p\varepsilon,$$

and analogously

$$\| v_{n,\varepsilon} \|_p \leqq (p\varepsilon)^{1/p}.$$

Since

$$(\alpha r - \beta s - \gamma t)^+ \geqq (\alpha r - \beta s - \max(\alpha, \gamma)\min(r, t))^+ \quad (r, s, t \in L_+^p, \alpha, \beta, \gamma \in \mathbf{R}_+)$$

$$\geqq (\alpha r - \max(\alpha, \beta)\cdot\max(r, s) - \max(\alpha, \gamma)\cdot\min(r, t))^+$$

we have

$$\begin{aligned}|T^n g_\varepsilon - T^{n+1}g_\varepsilon| &= |yT^n f_\varepsilon - (1+y)T^{n+1}f_\varepsilon + (1+x)T^{n+2}f_\varepsilon - xT^{n+3}f_\varepsilon| \\ &\geqq (yT^n f_\varepsilon - xT^{n+3}f_\varepsilon - (1+y)T^{n+2}f_\varepsilon)^+ \\ &\quad + ((1+x)T^{n+2}f_\varepsilon - (1+y)T^{n+1}f_\varepsilon - xT^{n+3}f_\varepsilon)^+ \\ &\quad + ((1+y)T^{n+1}f_\varepsilon - yT^n f_\varepsilon - (1+x)T^{n+2}f_\varepsilon)^+ \\ &\geqq (yT^n f_\varepsilon - xT^{n+3}f_\varepsilon - (1+y)v_{n,\varepsilon})^+ \\ &\quad + ((1+x)T^{n+2}f_\varepsilon - (1+y)u_{n+1,\varepsilon} - (1+x)u_{n+2,\varepsilon})^+ \\ &\quad + ((1+y)T^{n+1}f_\varepsilon - (1+y)u_{n,\varepsilon} - (1+y)u_{n+1,\varepsilon})^+\end{aligned}$$

and therefore

$$\begin{aligned}&\| T^n g_\varepsilon - T^{n+1}g_\varepsilon \|_p^p \\ &\geqq \| (yT^n f_\varepsilon - xT^{n+3}f_\varepsilon - (1+y)v_{n,\varepsilon})^+ \|_p^p \\ &\quad + \| ((1+x)(T^{n+2}f_\varepsilon - (1+y)u_{n+1,\varepsilon} - (1+x)u_{n+2,\varepsilon})^+ \|_p^p \\ &\quad + (1+y)^p \| (T^{n+1}f_\varepsilon - u_{n,\varepsilon} - u_{n+1,\varepsilon})^+ \|_p^p \\ &\geqq ((y \| T^n f_\varepsilon \|_p - x \| T^{n+3}f_\varepsilon \|_p - (1+y) \| v_{n,\varepsilon} \|_p)^+)^p \\ &\quad + ((1+x) \| T^{n+2}f_\varepsilon \|_p - (1+y) \| u_{n+1,\varepsilon} \|_p - (1+x) \| u_{n+2,\varepsilon} \|_p)^+)^p \\ &\quad + (1+y)^p((\| T^{n+1}f_\varepsilon \|_p - \| u_{n,\varepsilon} \|_p - \| u_{n+1,\varepsilon} \|_p)^+)^p \\ &\geqq ((y(1-\varepsilon) - x - (1+y)(p\varepsilon)^{1/p})^+)^p \\ &\quad + (((1+x)(1-\varepsilon) - (1+y)(p\varepsilon)^{1/p} - (1+x)(p\varepsilon)^{1/p})^+)^p \\ &\quad + (1+y)^p((1-\varepsilon - (p\varepsilon)^{1/p} - (p\varepsilon)^{1/p})^+)^p.\end{aligned}$$

Hence

$$\liminf_{\varepsilon\to 0} \inf_{n\in\mathbf{N}} \| T^n g_\varepsilon - T^{n+1}g_\varepsilon \|_p \geqq ((y-x)^p + (1+x)^p + (1-y)^p)^{1/p}. \tag{8}$$

Let us now estimate $\| g_\varepsilon \|_p$. Since $f_\varepsilon - v_{0,\varepsilon}$ and $T^2 f_\varepsilon$ have disjoint carrier we obtain

$$\begin{aligned}\| g_\varepsilon \|^p &\leqq (\| y(f_\varepsilon - v_{0,\varepsilon}) + xT^2 f_\varepsilon \|_p + y \| v_{0,\varepsilon} \|_p)^p + \| Tf_\varepsilon \|_p^p \\ &= ((y^p \| f_\varepsilon - v_{0,\varepsilon} \|_p^p + x^p \| T^2 f_\varepsilon \|_p^p)^{1/p} + y \| v_{0,\varepsilon} \|_p)^p + \| Tf_\varepsilon \|_p^p \\ &\leqq ((y^p + x^p)^{1/p} + y(p\varepsilon)^{1/p})^p + 1,\end{aligned}$$

$$\limsup_{\varepsilon\to 0} \| g_\varepsilon \|_p \leqq (y^p + x^p + 1)^{1/p}. \tag{9}$$

Now (8) and (9) together show that (iii) cannot hold if (7) is fulfilled. Thus, by contradiction, (iii) implies the existence of $\varepsilon' > 0$ such that

$$\inf_{n\in\mathbf{N}} \min(\| (T^n f - T^{n+1} f)^+ \|_p, \quad \| (T^n f - T^{n+2} f)^+ \|_p) \leqq 1 - \varepsilon' \tag{10}$$
$$(f \in L^p_+, \ \| f \|_p \leqq 1).$$

Next we want to deduce from (10) that there exists $\varepsilon'' > 0$ such that

$$\inf_{n\in\mathbf{N}} \| (T^n f - T^{n+2} f)^+ \|_p \leqq 1 - \varepsilon'' \qquad (f \in L^p_+, \ \| f \|_p \leqq 1). \tag{11}$$

To this end let $f \in L^p_+$, $\| f \|_p \leqq 1$ be given. If $\| (T^n f - T^{n+2} f)^+ \|_p \leqq 1 - \varepsilon'/2$ or $\inf_{n\in\mathbf{N}} \| T^n f \|_p \leqq 1 - \varepsilon'/4$ then any $0 < \varepsilon'' \leqq \varepsilon'/4$ will do it. In the other case, by (10), there exists $m_1 \in \mathbf{N}$ such that

$$\| (T^{m_1} f - T^{m_1+1} f)^+ \|_p \leqq 1 - \varepsilon'/2, \quad \inf_{n\in\mathbf{N}} \| T^n f \|_p \geqq 1 - \varepsilon'/4.$$

Denoting

$$u := \inf(T^{m_1} f, T^{m_1+1} f)$$

we have

$$| T^n u |^p + (T^n (T^{m_1} f - T^{m_1+1} f)^+)^p \leqq (T^n (u + (T^{m_1} f - T^{m_1+1} f)^+))^p$$
$$= | T^{m_1+n} f |^p,$$

$$\| T^n u \|_p^p \geqq \| T^{m_1+n} f \|_p^p - \| T^n (T^{m_1} f - T^{m_1+1} f)^+ \|_p^p \tag{12}$$
$$\geqq (1 - \varepsilon'/4)^p - (1 - \varepsilon'/2)^p \geqq \varepsilon'/4.$$

We now choose $m_2 \in \mathbf{N}$ so large such that

$$\inf_{n\in\mathbf{N}} \| T^{m_2+n} u \|_p \geqq (1 - \varepsilon'/4) \| T^{m_2} u \|_p. \tag{13}$$

By (10) there exists m_3 such that either

$$\| (T^{m_2+m_3} u - T^{m_2+m_3+2} u)^+ \|_p \leqq (1 - \varepsilon'/2) \| T^{m_2} u \|_p \tag{14}$$

or

$$\| (T^{m_2+m_3} u - T^{m_2+m_3+1} u)^+ \|_p \leqq (1 - \varepsilon'/2) \| T^{m_2} u \|_p. \tag{15}$$

In case (14) we denote

$$v := \inf(T^{m_2+m_3} u, T^{m_2+m_3+2} u)$$

and by now well-known arguments we have

$$\begin{aligned}\| v \|_p &\geqq \| T^{m_2+m_3}u \|_p - \| (T^{m_2+m_3}u - T^{m_2+m_3+2}u)^+ \|_p \\ &\geqq (1 - \varepsilon'/4 - 1 + \varepsilon'/2) \| T^{m_2}u \|_p \\ &\geqq (\varepsilon'/4)^2.\end{aligned}$$

Hence setting $m := m_1 + m_2 + m_3$ we have

$$\begin{aligned}\| (T^m f - T^{m+2}f)^+ \|_p^p \leqq \| T^m f - v \|_p^p \leqq \| T^m f \|_p^p - \| v \|_p^p &\leqq 1 - (\varepsilon'/4)^{2p} \\ &\leqq 1 - (\varepsilon'/4)^2.\end{aligned}$$

In case (15) we denote

$$\tilde{u} := \inf(T^{m_2+m_3}u, T^{m_2+m_3+1}u).$$

Replacing f by $\| T^{m_2}u \|_p^{-1} T^{m_2}u$ and u by $\| T^{m_2}u \|^{-1}\tilde{u}$ in (12) we get

$$\| T^n \tilde{u} \|_p^p \geqq \frac{\varepsilon'}{4} \| T^{m_2}u \|_p^p \geqq \left(\frac{\varepsilon'}{4}\right)^2 \qquad (n \in \mathbf{N}). \tag{16}$$

Sinc $T^n u \leqq \inf(T^{m_1+n}f, T^{m_1+n+1}f)$ we have

$$\tilde{u} \leqq \inf(T^m f, T^{m+1}f, T^{m+2}f), \qquad m := m_1 + m_2 + m_3.$$

In particular

$$\| (T^m f - T^{m+2}f)^+ \|_p^p \leqq \| T^m f - \tilde{u} \|_p^p \leqq \| T^m f \|_p^p - \| \tilde{u} \|_p^p \leqq 1 - (\varepsilon'/4)^2.$$

Hence, if we choose

$$0 < \varepsilon'' \leqq \min(\varepsilon'/4, 1 - (1 - (\varepsilon'/4)^2)^{1/p})$$

then (11) holds in all possible cases. By the already shown implication (iv)$\Rightarrow$(i), (10) implies

$$\lim_{n\to\infty} \| T^n f - T^{n+2} f \|_p = 0 \qquad (f \in L^p)$$

and the assertion (iii)$\Rightarrow$(iv) follows from the subsequent proposition, which itself is a consequence of (iv)$\Rightarrow$(i).

1.2. Proposition. *If*

$$\lim_{n\to\infty} \| T^n f - T^{n+2} f \|_p = 0 \qquad (f \in L^p)$$

and if there exists $\varepsilon > 0$ *such that*

$$\lim_{n\to\infty} \| T^n f - T^{n+1} f \|_p \leqq 2 - \varepsilon \qquad (f \in L^p,\ \| f \|_p \leqq 1)$$

then we have already

$$\lim_{n\to\infty} \| T^n f - T^{n+1} f \|_p = 0 \qquad (f \in L^p).$$

Proof. Assume that 1.1(iv) does not hold. Then for any $0 < \delta \leqq 1$ there exists $f_\delta \in L^p_+$, $\| f_\delta \| \leqq 1$ with

$$\lim_{n\to\mathbf{N}} \| (T^n f_\delta - T^{n+1} f_\delta)^+ \|_p \geqq (1-\delta)^{1/p}.$$

Setting

$$u_{n,\delta} := \inf(T^n f_\delta, T^{n+1} f_\delta), \qquad g_\delta := f_\delta - T f_\delta$$

we have

$$\| u_{n,\delta} \|_p^p \leqq \| T^n f_\delta \|_p^p - \| (T^n f_\delta - T^{n+1} f_\delta)^+ \|_p^p \leqq \delta$$

and therefore

$$\begin{aligned} \| T^n g_\delta - T^{n+1} g_\delta \|_p &= \| T^n f_\delta + T^{n+2} f_\delta - 2T^{n+1} f_\delta \|_p \\ &\geqq \| 2T^n f_\delta - 2T^{n+1} f_\delta \|_p - \| T^{n+2} f_\delta - T^n f_\delta \|_p \\ &= 2(\| T^n f_\delta - u_{n,\delta} \|_p^p + \| T^{n+1} f_\delta - u_{n,\delta} \|^p)^{1/p} \\ &\quad - \| T^{n+2} f_\delta - T^n f_\delta \|_p \\ &\geqq 2((\| T^n f_\delta \|_p - \| u_{n,\delta} \|_p)^p \\ &\qquad + (\| T^{n+1} f_\delta \|_p - \| u_{n,\delta} \|)^p)^{1/p} \\ &\quad - \| T^{n+2} f_\delta - T^n f_\delta \|_p . \end{aligned}$$

Hence for any $0 < \delta < 1$ there exists $n_\delta \in \mathbf{N}$ such that

$$\| T^n g_\delta - T^{n+1} g_\delta \|_p \geqq 2((1-\delta^{1/p})^p + (1-\delta^{1/p})^p)^{1/p} - \delta \qquad (n \geqq n_\delta)$$

contradicting

$$\limsup_{n\to\infty} \| T^n g - T^{n+1} g \|_p \leqq (2-\varepsilon) \| g \|_p \qquad (g \in L^p).$$

It may very well happen that $\| T^n f - T^{n+1} f \|_p = 2$ for an $f \in L^p$, $\| f \| = 1$. On the other hand, the next example shows that condition 1.1(iv) cannot be weakened. For an operator S on L^p, $|||S|||_p$ will be its operator norm.

1.3. Example. Let $\Omega := \{1, 2, 3\}$ be endowed with the counting measure μ and $\tau: \Omega \to \Omega$ be defined by

$$\tau(1) = 2, \quad \tau(2) = 3, \quad \tau(3) = 1.$$

Then

$$Tf := f \circ \tau$$

defines an isometry in $L^p(\mu)$ for any $1 \leqq p \leqq \infty$. Hence if

$$\lim_{n \to \infty} \| T^n f - T^{n+1} f \|_p = 0$$

then $f = Tf$ and therefore $f = 1$.

In order to calculate the operator norm $|||T^n - T^{n+1}|||_p = |||I - T|||_p$ we identify functions on Ω with vectors in $\mathbf{R}^3$. We have

$$\| (x, y, z) - T(x, y, z) \|_p = (|x - y|^p + |y - z|^p + |x - z|^p)^{1/p} \tag{1}$$
$$(1 \leqq p < \infty).$$

Choosing $z = -1$, $0 \leqq x \leqq y$ we see that $|||I - T|||_p \geqq \alpha_p$.

Let now (x, y, z), $\| (x, y, z) \|_p = 1$ be such that

$$\| (x, y, z) - T(x, y, z) \|_p = |||I - T|||_p.$$

By (1) x, y, z cannot all have the same sign. Since the right hand side of (1) is symmetric in x, y, z we may assume that $z \leqq 0 \leqq x \leqq y$. If $1 < p < \infty$ then we must have $z < 0$, otherwise

$$\| (I - T)(x, y, z) \|_p = (|x - y|^p + 1)^{1/p} \leqq 2^{1/p} < (1 + 2^{p-1})^{1/p}$$
$$= (I - T)(2^{1/p}, -2^{1/p}, 0).$$

Setting $x' := x/-z$, $y' = y/-z$ we obtain

$$|||I - T|||_p = |z|^{-1}(|x' - y'|^p + |y' + 1|^p + |x' + 1|^p)^{1/p}$$
$$= \left(\frac{|x' + 1|^p + |y' + 1|^p + |x' - y'|^p}{1 + |x'|^p + |y'|^p} \right)^{1/p} \leqq \alpha p$$

provided $1 < p < \infty$. If $p = 1$ then $\alpha_p = 2$ and $|||I - T|||_p \leqq \alpha_p$ is trivial. Since the adjoint operator T^* is of the same type, i.e.

$$T^* f = f \circ \tau^{-1},$$

we have also

$$|||I - T^*|||_p = \alpha_p \qquad (1 \leqq p < \infty).$$

Hence

$$|||I - T|||_\infty = |||I - T^*|||_1 = 2.$$

1.4. *Estimates for* α_p. We have already seen in the proof of Theorem 1.1 that

(a) $(1 + 2^{p-1})^{1/p} \leqq \alpha_p$.

Let T be the operator of Example 1.3. Then

$$\alpha_p = |||I - T|||_p = |||I - T^*|||_q = \alpha_q$$

implies

(b) $\alpha_p = \alpha_q \; (1 < p, \, q < \infty, \; 1/p + 1/q = 1)$.

By the interpoation theorem of M. Riesz [9] we have

(c) $p \to \log \alpha_p$ is a convex function on $[1, \infty[$. Note that Riesz's original theorem is valid also for real L^p space, whereas Thorin's generalization holds only for complex L^p spaces.

Since $I - T$ is a normal operator on L^2 we have

$$|||I - T|||_2 = \sup\{|\lambda| : \lambda \in \mathbf{C} \text{ eigen value of } I - T\}.$$

This is well known for the operator norm on the complexification of L^2, but for $p = 2$ the complex and the real operator norm are equal. Obviously T has the eigen values

$$1, \quad e^{2i\pi/3}, \quad e^{4i\pi/3}$$

and therefore $I - T$ has the eigen values

$$0, \quad 1 - e^{2i\pi/3}, \quad 1 - e^{4i\pi/3}.$$

Finally we get

(d) $\alpha_2 = |1 - e^{2i\pi/3}| = \sqrt{3}$.

From (b), (c), (d) one gets an upper estimate for α_p. In particular we have

(e) $\alpha_p < 2 \; (1 < p < \infty)$.

Since, for any $f \in L^p_+$, $\| T^n f - T^{n+1} f \|_p^p \leqq \| T^n f \|_p^p \leqq 2 \| f \|_p^p$ we obtain the following "0–$2^{1/p}$" law.

1.5. Corollary.

$$\sup_{f \in L^p_+, \|f\|_p \leqq 1} \lim_{n \to \infty} \| T^n f - T^{n+1} f \| \in \{0, 2^{1/p}\}.$$

1.6. Corollary. *If one of the conditions in Theorem* 1.1 *holds, then for* *any* $f \in L^p$ *the* L^p*-convergence of* $A_n f := (1/n) \sum_{i=0}^{n-1} T^i f$ *implies the* L^p*-convergence of* $T^n f$ *(both limits are of course equal). If* $p > 1$ *then* $A_n f$ *converges always in* L^p *by the mean ergodic theorem.*

PROOF. By Krengel [5], p. 73 we have the decomposition

$$M = \bar{N} + F$$

where

$$M := \left\{ f \in L^p : \lim_{n \to \infty} A_n f \text{ exists} \right\},$$

$$N := \{ f - Tf : f \in L^p \},$$

$$F := \{ f \in L^p : Tf = f \}.$$

Hence we need to prove our assertion only for $f \in N$ and $f \in F$. For $f \in F$ this is trivial and for $f \in N$ this follows from Theorem 1.1.

We have the following uniform variant of Theorem 1.1 and Corollary 1.6.

1.7. THEOREM. *The following conditions are equivalent:*

(i) $\lim_{n \to \infty} ||| T^n - T^{n+1} |||_p = 0$.

(ii) *There exists* $\varepsilon > 0$ *and* $n \in \mathbf{N}$ *such that*

$$\| T^n f - T^{n+1} f \| \leqq (1 - \varepsilon) 2^{1/p} \qquad (f \in L^p_+,\ \| f \|_p \leqq 1).$$

(iii) $\lim_{n \to \infty} ||| T^n - T^{n+1} |||_p < \alpha_p$.

(iv) $\inf_{n \in \mathbf{N}} \sup_{f \in L^p_+, \| f \|_p \leqq 1} \| (T^n f - T^{n+1} f)^+ \|_p < 1$.

If T *is also uniformly ergodic* (cf. Krengel [5], p. 86) *and one of the above conditions is fulfilled, then there exists a projector* $P: L^p \to L^p$ *with* $||| P |||_p \leqq 1$ *and*

$$\lim_{n \to \infty} ||| T^n - P |||_p = 0.$$

PROOF. The proof of Theorem 1.1 can be made "uniform" by experienced readers. For more details the reader is referred to our forthcoming paper where the uniform result 1.10(b) is proved.

For the proof of the second assertion we note first that $N := \{ f - Tf : f \in L^p \}$ is closed in L^p by Krengel [5], p. 87. We can write

$$L^p = N \oplus F, \quad F := \{ f \in L^p : Tf = f \}$$

by Krengel [4], p. 73. Thus the operator $(I - T)$ is a continuous bijection of the Banach space N into itself. Hence by the open mapping theorem there exists a bounded operator $S\colon N \to N$ such that $(I - T)S = I$. Using now 1.7(i) we get

$$\limsup_{n\to\infty} \{ \| T^n g \|_p : g \in N,\ \| g \|_p \leqq 1\}$$

$$= \limsup_{n\to\infty} \{ \| (T^n - T^{n+1})Sg \|_p : g \in N,\ \| g \|_p \leqq 1\}$$

$$= 0.$$

Recalling the decomposition $L^p = N \oplus F$ the assertion with $P = \lim_{n\to\infty} (1/n) \sum_{i=1}^{n-1} T^i$ is now immediate.

If T is also compact (quasi-compactness would be enough), then much more than Theorem 1.1 and Corollary 1.6 is true:

1.8. Corollary. *Assume that T^{n_0} is also a compact operator for an $n_0 \in \mathbf{N}$. Then the folowing conditions are equivalent*:

(i) $\lim_{n\to\infty} ||| T^n - T^{n+1} |||_p = 0$.

(ii) $\lim_{n\to\infty} \| T^n f - T^{n+1} f \|_p < 2^{1/p}$ $(f \in L^p_+,\ \| f \|_p \leqq 1)$.

(iii) $\lim_{n\to\infty} \| T^n f - T^{n+1} f \|_p < \alpha_p$ $(f \in L^p,\ \| f \|_p \leqq 1)$.

Moreover, if one of the above conditions is fulfilled then there exists a projector $P\colon L^p \to L^p$ with finite dimensional range, $||| P |||_p \leqq 1$ and

$$\lim_{n\to\infty} ||| T^n - P |||_p = 0.$$

Proof. Let X (resp. X_+) be the closure in L^p of

$$\{T^{n_0} f\colon f \in L^p,\ \| f \|_p \leqq 1\} \quad (\text{resp. } \{T^{n_0} f\colon f \in L^p_+,\ \| f \|_p \leqq 1\}).$$

By our assumption both sets are compact and the functions

$$\varphi_n(f) := \| (T^n - T^{n+1}) f \|_p$$

are continuous on X and X_+. The sequence (φ_n) is also decreasing. Hence $\varphi := \inf_n \varphi_n$ is upper semi-continuous on the compact sets X, X_+. If now (iii) holds then $\sup \varphi(f)\colon f \in X\} < \alpha_p$. Furthermore $\{\varphi_n < (1 + 2^{p-1})^{1/p}\}$ defines an increasing sequence of open sets covering X.

By compactness there exists $n \in \mathbf{N}$ such that $\varphi_n < \alpha_p$. This means exactly

$$||| T^{n+n_0} - T^{n+n_0+1} ||| < \alpha_p.$$

Hence 1.7(iii) is satisfied. Replacing (iii) by (ii) and X by X_+ one can see by the same method that (ii) implies 1.7(ii). Since T is also uniformly ergodic by Krengel [5], p. 89, the assertion follows from Theorem 1.7.

Finally we want to show that a recent uniform "0–2" law of Zaharopol [11] for $1 < p < \infty$, $p \neq 2$ and of Katznelson and Ttzafriri [4] in the general case follows from Theorem 1.7. To this end we recall that the linear space of all differences of positive operators in L^p is a vector lattice under its canonical order (cf. Schaefer [10], p. 229). In particular the infimum $T_1 \wedge T_2$ and the modulus $|T_1 - T_2| := T_1 + T_2 - 2T_1 \wedge T_2$ of two positive operators exist.

1.9. Corollary. *If* $\limsup_{n\to\infty} ||| \, |T^n - T^{n+1}| \, |||_p < 2$, *then we have* $\lim_{n\to\infty} ||| T^n - T^{n+1} |||_p = 0$.

Proof. By Zaharopol [11], Proposition 4.2 there exists $\delta > 0$ such that, for any $f \in L_p$ with $\| f \|_p \leqq 1$, either

(1) $\| T^{n+1} f \|_p \leqq 1 - \delta/2$

or

(2) $\| T^{n+1} f \|_p \geqq 1 - \delta/2$, $\| (T^{n+1} - T^n \wedge T^{n+1}) f \| \leqq 1 - \delta$

holds. In case (1) we have

$$\begin{aligned} \| T^n f - T^{n+1} f \|_p^p &\leqq \| (T^n f - T^{n+1} f)^+ \|_p^p + \| (T^{n+1} f - f - T^n f)^+ \|_p^p \\ &\leqq 1 + (1 - \delta/2)^p. \end{aligned}$$

In case (2) we put $u := \inf(T^n f, T^{n+1} f) \leqq (T^n \wedge T^{n+1}) f$. Then

$$\| u \|_p \geqq \| T^{n+1} f \|_p - \| (T^{n+1} - T^n \wedge T^{n+1}) f \|_p \geqq 1 - \delta/2 - (1 - \delta) = \delta/2$$

implies

$$\begin{aligned} \| T^n f - T^{n+1} f \|_p^p &= \| T^n f - u \|_p^p + \| T^{n+1} f - u \|_p^p \\ &\leqq \| T^n f \|_p^p - \| u \|_p^p + \| T^{n+1} f \|_p^p - \| u \|_p^p \\ &\leqq 2(1 - (\delta/2)^p). \end{aligned}$$

Thus condition 1.7(ii) holds in all cases.

1.10. Remarks. (a) In a forthcoming paper we will show a non-uniform variant of Corollary 1.9, i.e. the existence of an $\varepsilon > 0$ with

$$\limsup_{n\to\infty} \| \, |T^n - T^{n+1}| f \|_p < 2 - \varepsilon \qquad (f \in L^p)$$

implies already

$$\lim_{n\to\infty} \| (T^n - T^{n+1}) f \|_p = 0.$$

This result cannot be shown by the method of Katznelson and Tzafriri [4].

(b) In that paper we will also show that

$$\limsup_{n\to\infty} ||| \, |T^n - T^{n+1}| \, |||_p < 2$$

implies the stronger conclusion

$$\lim_{n\to\infty} ||| \, |T^n - T^{n+1}| \, |||_p = 0.$$

2. Positive contraction in $C(X)$ and L^∞

In this section B will be either the Banach space of continuous functions on a compact space X or the space L^∞ with respect to a σ-finite measure space. We denote by B_+ the cone of non-negative elements. Of course, the L^∞ case is a special case of the $C(X)$ case, by the Stone representation, but the point is that the proofs can be done directly in L^∞.

2.1. THEOREM. *For a positive, linear contraction $T: B \to B$ the following conditions are equivalent*:

(i) $\lim_{n\to\infty} \| T^n f - T^{n+1} f \| = 0 \ (f \in B)$.

(ii) *There exists $\varepsilon > 0$ such that*

$$\lim_{n\to\infty} \| T^n f - T^{n+1} f \| < 1 - \varepsilon \qquad (f \in B_+, \ \| f \| \leqq 1).$$

(iii) *There exists $\varepsilon > 0$ such that*

$$\lim_{n\to\infty} \| T^n 1 - T^{n+1} 1 \| < 2\varepsilon$$

and

$$\lim_{n\to\infty} \| T^n f - T^{n+1} f \| < 2(1-\varepsilon) \qquad (f \in B, \ \| f \| \leqq 1).$$

PROOF. (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii) are obvious. Assume now that (ii) holds and let $f \in B_+$, $\| f \| \leqq 1$ be given. Then there exists $n_1 \in \mathbb{N}$ such that

$$\| (T^{n_1} f - T^{n_1+1} f)^+ \| \leqq 1 - \varepsilon, \quad \| (T^{n_1-1} f - T^{n_1} f)^+ \| \leqq 1 - \varepsilon.$$

For $h := \inf(\varepsilon, T^{n_1} f, T^{n_1+1} f)$ we have

$$\| h \| \leqq \varepsilon, \quad \| T^{n_1} f - h \| \leqq 1 - \varepsilon, \quad \| T^{n_1+1} f - h \| \leqq 1 - \varepsilon.$$

Hence, putting

$$u_1 := \tfrac{1}{2} h, \quad v_1 := \tfrac{1}{2}(T(T^{n_1} f - h) + T^{n_1+1} f - h)$$

we get

$$T^{n_1+1}f = \tfrac{1}{2}(Th + T(T^{n_1}f - h) + h + (T^{n_1+1}f - h)) = u_1 + Tu_1 + v_1$$

and

$$\| u_1 \| \leqq \varepsilon/2, \quad \| v_1 \| \leqq \tfrac{1}{2}(\| T^{n_1}f - h \| + \| T^{n_1+1}f - h \|) \leqq 1 - \varepsilon.$$

From now on the proof is essentially the same as in [8]. Replacing f by v_1 the above construction yields $\tilde{u}_2$, $v_2 \in B_+$, $\tilde{n}_2 \in \mathbf{N}$ such that

$$T^{\tilde{n}_2+1}v_1 = \tilde{u}_2 + T\tilde{u}_2 + v_2, \quad \| \tilde{u}_2 \| \leqq \tfrac{1}{2}\varepsilon \| v_1 \|, \quad \| v_2 \| \leqq (1-\varepsilon) \| v_1 \|.$$

Defining

$$n_2 := n_1 + \tilde{n}_2 + 1, \quad u_2 := \tilde{u}_2 + T^{\tilde{n}_2+1}u_1$$

we have

$$T^{n_2+1}f = u_2 + Tu_2 + v_2, \quad \| u_2 \| \leqq \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon(1-\varepsilon), \quad \| v_2 \| \leqq (1-\varepsilon)^2.$$

Continuing in this way we can construct sequences (u_i), (v_i) in B_+ and (n_i) such that

$$T^{n_i+1}f = u_i + Tu_i + v_i, \quad \| v_i \| \leqq (1-\varepsilon)^i,$$

$$u_i \leqq \frac{\varepsilon}{2} \sum_{j=0}^{i-1} (1-\varepsilon)^j \leqq \tfrac{1}{2}.$$

Let now $\delta > 0$ be given. Choosing i so large such that $(1-\varepsilon)^i \leqq \delta$ and setting

$$m_1 := n_i + 1, \quad r_1 := u_i, \quad s_1 := v_i$$

we have the representation

$$T^{m_1}f = r_1 + Tr_1 + s_1, \quad \| r_1 \| \leqq \tfrac{1}{2}, \quad \| s_1 \| \leqq \delta.$$

Replacing f by r_1 we can find $\tilde{m}_2 \in \mathbf{N}$, r_2, $\tilde{s}_2 \in B_+$ with

$$T^{\tilde{m}_2}r_1 = r_2 + Tr_2 + \tilde{s}_2, \quad \| r_2 \| \leqq \tfrac{1}{4}, \quad \| \tilde{s}_1 \| \leqq \tfrac{1}{2}\delta.$$

Defining $m_2 := \tilde{m}_1 + m_2$, $s_2 := T^{\tilde{m}_2}s_1 + (I+T)\tilde{s}_2$,

$$T^{m_2}f = (I+T)^2 r_2 + s_2, \quad \| r_2 \| \leqq \tfrac{1}{4}, \quad \| s_2 \| \leqq 2\delta.$$

Continuing in this way we can construct (r_k), (s_k) in B_+ and (m_k) such that

$$T^{m_k}f = (I+T)^k r_k + s_k, \quad \| r_k \| \leqq 2^{-k}, \quad \| s_k \| \leqq k\delta.$$

In particular, for even $k \in \mathbf{N}$ we have

$$\begin{aligned} \| T^{m_k} f - T^{m_k+1} f \| &= \| (I-T)(I+T)^k r_k + s_k \| \\ &\leqq \| s_k \| + \| r_k \| + \| T^{k+1} r_k \| \\ &\quad + \sum_{i=1}^{k} \left| \binom{k}{i} - \binom{k}{i-1} \right| \, \| T^i r_k \| \\ &\leqq k\delta + 2^{1-k} + 2^{-k} 2 \binom{k}{k/2} \end{aligned}$$

since $\binom{k}{i} - \binom{k}{i-1}$ is increasing on $1 \leqq i \leqq k/2$ and decreasing on $k/2 \leqq i \leqq k$. Choosing $\delta > 0$ sufficiently small we can find for any even $k \in \mathbf{N}$ an $n \in \mathbf{N}$ such that

$$\| T^n f - T^{n+1} f \| \leqq 2^{3-k} \binom{k}{k/2}.$$

By Stirling's formula the expression on the right side tends to 0 and (i) follows.

Assume now that (iii) holds and let $0 \leqq g \leqq 1$ be given. Putting $f := g - 1$ we have

$$\| f \| \leqq 1, \quad g = \tfrac{1}{2} f + \tfrac{1}{2}.$$

Applying (iii) to f we get

$$\begin{aligned} \lim_{n \to \infty} \| T^n g - T^{n+1} g \| &\leqq \lim_{n \to \infty} \tfrac{1}{2} \| T^n f - T^{n+1} f \| + \lim_{n \to \infty} \tfrac{1}{2} \| T^n 1 - T^{n+1} 1 \| \\ &\leqq 1 - \varepsilon' \end{aligned}$$

where

$$\varepsilon' := \varepsilon - \lim_{n \to \infty} \tfrac{1}{2} \| T^n 1 - T^{n+1} 1 \| > 0.$$

Hence (ii) is satisfied if we replace ε by the above ε'.

2.2. REMARKS. (a) It is an open question whether the condition

$$\lim_{n \to \infty} \| T^n 1 - T^{n+1} 1 \| < \varepsilon$$

can be removed.

(b) Analogues of Corollary 1.6, Theorem 1.7, Corollary 1.8 hold also in the present setting with no change of proof, since the special properties of L^p-norms were not used there.

(c) It is a natural question whether Theorem 2.1 holds for general Banach lattices. We have no counterexample. Under additional conditions on T Greiner and Nagel [3] have given a different kind of "zero–two" law for very general Banach lattices. Recently H. H. Schaefer (private communication) generalized the Zaharopol–Katznelson–Tzafriri Theorem (Corollary 1.9) to arbitrary Banach lattices.

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